Discrete dynamical models: combinatorics, statistics and continuum approximations

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Abstract. This essay advocates the view that any problem that has a meaningful empirical content, can be formulated in constructive, more definitely, finite terms. We consider combinatorial models of dynamical systems and approaches to statistical description of such models. We demonstrate that many concepts of continuous physics — such as continuous symmetries, the principle of least action, Lagrangians, deterministic evolution equations — can be obtained from combinatorial structures as a result of the large number approximation. We propose a constructive description of quantum behavior that provides, in particular, a natural explanation of appearance of complex numbers in the formalism of quantum mechanics. Some approaches to construction of discrete models of quantum evolution that involve gauge connections are discussed.

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1. Introduction

Any continuous physical model is empirically equivalent to a certain finite model. This thesis is widely used in practice: solution of differential equations by the finite difference method or by using truncated series is typical example.

It is often believed that continuous models are "more fundamental" than discrete or finite ones. However, there are many indications that nature is fundamentally discrete at small (Planck) scales, and is possibly finite.[1] Moreover, description of physical systems by, e.g., differential equations can not be fundamental in principle, since it is based on approximations of the form $f(x) \approx f(x_0) + \nabla f(x_0)\Delta x$.

This essay advocates the view that finite models provide a more relevant description of physical reality than continuous models which are only approximations in the limit of large numbers.[2] Using simple combinatorial models, we show how such concepts as continuous symmetries, the principle of least action, Lagrangians, deterministic evolution equations, etc. arise from combinatorial structures as a result of the large number approximation. We also consider some approaches to the construction of discrete models of quantum behavior and related models describing evolution of gauge connections.

Any statistical description assumes one or another concept of *macrostate*. We define macrostates as equivalence classes of microstates. This definition is especially convenient for models incorporating symmetry groups. We distinguish two types of statistical models:

1. *Isolated system* is purely combinatorial object in the sense that "probability" of a microstate has *a priori* nature. Namely, all microstates are equiprobable, so their probabilities are equal to the inverse of their total number. The macrostates are specified by an equivalence relation on microstates.

2. *Open system* is obtained from an isolated system by the following modification: The macrostates are specified by the same equivalence relation, but the probabilities of microstates depend on some parameters, which are introduced for approximate description of interaction of the system with the environment.

The archetypal examples of isolated and open systems are, respectively, *microcanonical* (macrostates are defined as collections of microstates with equal energies) and *canonical* (macrostates are defined similarly, and interaction with the environment is parameterized by the temperature) *ensembles*.

The classical description of a (reversible) dynamical system looks schematically as follows. There are a set X of states and a group $G_{\text{cl}} \leq \text{Sym}(X)$ of transformations (bijections) of X. Evolutions of X are described by sequences of group

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[1]The total number of binary degrees of freedom in the Universe is about 10^{122} as estimated via the holographic principle and the Bekenstein–Hawking formula.

[2]Comparing the Planck length, $\sim 10^{-35}$ m, with the minimum length observable in experiment, $\sim 10^{-15}$ m, we may assume that the emergence of the empirically perceived continuous space is provided by averaging over about 10^{20} discrete elements.

elements $g_t \in G_{\text{cl}}$ parameterized by the *continuous* time $t \in \mathcal{T} = [t_a, t_b] \subseteq \mathbb{R}$. The observables are functions $h : X \to \mathbb{R}$.

We can "quantize" an arbitrary set X by assigning numbers from a number system $\mathcal{F}$ to the elements $x \in X$, i.e., by interpreting X as a basis of a module. The quantum description of a dynamical system assumes that the module associated with the set of classical states X is a Hilbert space $\mathcal{H}_X$ over the field of complex numbers, i.e., $\mathcal{F} = \mathbb{C}$; the transformations g_t and observables h are replaced by unitary U_t and Hermitian H operators on $\mathcal{H}_X$, respectively.

To make the quantum description constructive, we propose the following modifications. We assume that the set X is finite. Operators U_t belong to the group of unitary transformations of the Hilbert space $\mathrm{Aut}\,(\mathcal{H}_X)$. Using the fact that this group contains a finitely generated — and hence residually finite — dense subgroup, we can replace $\mathrm{Aut}\,(\mathcal{H}_X)$ by unitary representation of some finite group G that is suitable to provide an empirically equivalent description of a particular problem. A plausible assumption about the nature of quantum amplitudes implies that the field $\mathbb{C}$ can be replaced by an abelian number field[1] $\mathcal{F}$. This field is a subfield of a certain *cyclotomic* field $\mathbb{Q}_m$ which in turn is a subfield of the complex field: $\mathcal{F} \leq \mathbb{Q}_m < \mathbb{C}$. The natural number m, called *conductor*, is determined by the structure of the group G. Note that the fields $\mathbb{Q}_m$ and $\mathbb{C}$ provide empirically equivalent descriptions in any applications, because $\mathbb{Q}_m$ is a dense subfield of $\mathbb{C}$ for any $m \geq 3$.

In this paper we will assume that the time $\mathcal{T}$ is *discrete* and can be represented as a sequence of integers, typically $\mathcal{T} = [0, 1, \ldots, T]$.

Note also that the subscript a in the notation $\mathcal{H}_a$ for Hilbert spaces is overloaded and can mean, depending on context: dimension of the space, a set on which the space is spanned, a group whose representation space is $\mathcal{H}_a$, etc.

2. Scheme of statistical description

For convenience of presentation, let us fix some notation:

- U is the *full set of states* of a system A. The states from U are usually called *"microstates"* in statistical mechanics.

- $N = |U|$ is the total number of microstates.

- p_u is the *probability (weight)* of a microstate $u \in U$, $\sum_{u \in U} p_u = 1$.

- $\sim$ is an *equivalence relation* on the set U.

- Taking an element $u \in U$, we define the *macrostate* λ as the equivalence class $\lambda = \{v \in U \mid v \sim u\}$.

- Λ denotes the set of macrostates.

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[1] An abelian number field is an algebraic extension of $\mathbb{Q}$ with abelian Galois group. According to the Kronecker–Weber theorem, any such extension is contained in some cyclotomic field.

- The equivalence relation $\sim$ determines the *partition* $U = \coprod_{\lambda \in \Lambda} \lambda$.

- $K = |\Lambda|$ is the number of macrostates.

- $N_\lambda = |\lambda|$ is the size of a macrostate $\lambda \in \Lambda$.

- P_λ denotes the probability of an arbitrary microstate from U to belong to the macrostate λ.

Isolated systems. The probability of a microstate of an isolated system is defined naturally[1] as $p_u = 1/N$ for any $u \in U$, and the probability of any microstate from U to belong to the macrostate λ is, respectively, $P_\lambda = N_\lambda/N$. Since the "probabilities" in isolated systems have an *a priori* nature, such systems are in fact purely combinatorial objects, and we can talk about the number of combinations instead of probability.

Open systems interact with the environment. This interaction is parameterized by assigning, in accordance with some rule, probabilities to all microstates. That is, the probability of a microstate $u \in \lambda$ is a function

$$p_u = p_u(\alpha_1, \alpha_2, \ldots) \tag{1}$$

of some parameters $\alpha_1, \alpha_2, \ldots$. These parameters and function (1) are determined by the specifics of a particular problem. For open systems $P_\lambda = \sum_{u \in \lambda} p_u(\alpha_1, \alpha_2, \ldots)$.

Entropy. One of the central issues of the statistical description is the search for the most probable macrostates, i.e. the macrostates with the maximum value of P_λ. Technically, entropy is defined as the logarithm of the number (or probability) of microstates that belong to a particular macrostate. The concept of entropy is convenient for two reasons:

1. Since the logarithm is a monotonic function, the logarithm of any function has the same extrema as the function itself.

2. If a system can be represented as a combination of two independent systems, $A = A' \otimes A''$, then the macrostates of A can be represented as $\lambda' \otimes \lambda''$. So, when computing entropy of such decompositions, we can replace multiplication by a simpler operation — addition: $\log(P_{\lambda'} P_{\lambda''}) = \log P_{\lambda'} + \log P_{\lambda''}$.

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[1]This is "the equal *a priori* probability postulate" of statistical mechanics [1].

Stirling's formula is one of the main tools for obtaining continuum approximations of combinatorial expressions:

$$\ln n! \approx \underbrace{n \ln n - n}_{\text{superlinear}} + \underbrace{\frac{1}{2} \ln(2\pi n)}_{\text{logarithmic}} + \underbrace{\frac{1}{12n} - \frac{1}{360 n^3} + \cdots}_{\text{decreasing terms}} \tag{2}$$

For our purposes it is sufficient to retain only the terms which grow with n, i.e. the *superlinear* and *logarithmic* terms.

2.1 Examples of isolated and open systems

To illustrate the above, let us give a few examples of isolated and open systems.

Sequences of symbols. Isolated system. Let Σ be an alphabet of size M:

$$\Sigma = (\sigma_1, \ldots, \sigma_m, \ldots, \sigma_M). \tag{3}$$

The microstates are sequences of the length T of symbols from Σ:

$$u = a_1 \cdots a_t \cdots a_T \in U, \quad a_t \in \Sigma.$$

The total number of microstates is $N = M^T$. Let $k^u = (k_1^u, \ldots, k_m^u, \ldots, k_M^u)$ be the vector of multiplicities of symbols σ_m in a microstate u. It is obvious that $k_1^u + \cdots + k_M^u = T$. We define the equivalence relation $\sim$ as follows

$$u \sim v \iff k^u = k^v \equiv k. \tag{4}$$

The macrostate λ_k, defined by equivalence (4), consists of all sequences with the multiplicity vector k. The total number of macrostates is

$$K = \binom{T + M - 1}{M - 1}.$$

The size of the macrostate λ_k is

$$N_{\lambda_k} = \frac{T!}{k_1! k_2! \cdots k_M!}.$$

Introducing the vector of "frequencies" $(f_1 = k_1/T, \ldots, f_M = k_M/T)$ and applying the leading part of Stirling approximation $\log n! \approx n \log n - n$ to the entropy $S_{\lambda_k} = \log N_{\lambda_k}$ of the macrostate λ_k we obtain $S_{\lambda_k} \approx T H(X)$, where

$$H(X) = -\sum_{i=1}^{M} f_i \log f_i$$

is the *Shannon entropy* [2] of a random variable X whose M outcomes have probabilities $f_1, \ldots, f_M$.

The model of symmetric random walk [3] is a slight modification of the above isolated system. Alphabet (3) contains now an even number of symbols $M = 2d$, and it is divided into two parts $\Sigma = \Sigma_+ \coprod \Sigma_-$, where

$$\Sigma_+ = \{\sigma_i \in \Sigma \mid 1 \leq i \leq d\} \quad \text{and} \quad \Sigma_- = \{\sigma_i \in \Sigma \mid d+1 \leq i \leq M\}.$$

The elements of Σ_+ and Σ_- can be interpreted, respectively, as "positive" and "negative" unit steps in the d directions of the integer lattice $\mathbb{Z}^d$.

The vector of multiplicities of symbols from Σ can be written as $k = (k_+, k_-)$, where $k_+ = (k_1, \ldots, k_d)$ and $k_- = (k_{d+1}, \ldots, k_{2d})$. The equivalence of microstates u and v is defined now as follows

$$u \sim v \iff k_+^u - k_-^u = k_+^v - k_-^v. \tag{5}$$

The partition defined by (5) is a *coarsening*[1] of the partition defined by (4). The numbers of equivalence classes are *figurate numbers* of some d-dimensional regular convex polytopes — d-dimensional analogues of the octahedron. For example, in the case $d = 3$ the number of macrostates is equal to the $(T + 1)$th *octahedral number*:

$$K = (T + 1) \frac{2(T + 1)^2 + 1}{3}.$$

Multinomial distribution. Open system. The microstates are also sequences of symbols from alphabet (3). But now the presence of an environment is assumed. The influence of the environment is parameterized by the assumption that any symbol $\sigma_m \in \Sigma$ comes with a fixed individual probability α_m, such that $\sum_{m=1}^{M} \alpha_m = 1$. Thus, $\alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_M^{k_M} \equiv p_{\lambda_k}$ is the probability of a microstate from a macrostate λ_k, defined by equivalence relation (4). The probability of a microstate from U to belong to the macrostate λ_k is described by the *multinomial distribution*:

$$P_{\lambda_k} = N_{\lambda_k} p_{\lambda_k} = \frac{T!}{k_1! k_2! \cdots k_M!} \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_M^{k_M}. \tag{6}$$

Microcanonical ensemble. Isolated system. The concept of a microcanonical ensemble is based on the classification of microstates by energy. More specifically, if there is a real-valued function on microstates $E : U \to \mathbb{R}$, then we can impose the equivalence relation on U:

$$u \sim v \iff E(u) = E(v) \equiv \mathrm{E}, \quad u, v \in U. \tag{7}$$

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[1]The partition $\mathcal{P}_2 = \{B_i\}$ of a set S is a *coarsening* of the partition $\mathcal{P}_1 = \{A_j\}$ of the same set if for every subset $A_j \in \mathcal{P}_1$ there is a subset $B_i \in \mathcal{P}_2$ such that $A_j \subseteq B_i$. The opposite relation among partitions is called the *refinement* [2].

Assuming that the *energy* E takes a finite number of values: $E \in \{E_1, E_2, \ldots, E_K\}$, we define the *microcanonical ensemble* as the macrostate λ_k which is an equivalence class of relation (7), i.e., the set of microstates with the energy E_k. More formally:

$$\lambda_k = \{u \mid u \in U \wedge E(u) = E_k\} \ . \tag{8}$$

In statistical mechanics [4] the microcanonical ensemble is defined in terms of the *Boltzmann entropy* formula

$$S_{\lambda_k} = k_B \ln N_{\lambda_k}$$

or, equivalently, via the *microcanonical partition function*

$$N_{\lambda_k} = e^{S_{\lambda_k}/k_B},$$

where $N_{\lambda_k} = |\lambda_k|$ and k_B is the Boltzmann constant (we may assume that $k_B = 1$).

Canonical ensemble. Open system. A *canonical ensemble* is an open counterpart of the microcanonical ensemble. Macrostates of the canonical ensemble are defined by (8). Probability (1), that parameterizes the interaction with the environment in our general scheme, is now a function of a single parameter $\alpha = T$, the *temperature* of the environment. Namely, the probability of a microstate $u \in \lambda_k$ is given by the *Gibbs formula*

$$p_{\lambda_k}(T) = \frac{1}{Z} e^{-E_k/k_B T},$$

where the normalization constant

$$Z = \sum_{k=1}^{K} N_{\lambda_k} e^{-E_k/k_B T}$$

is called the *canonical partition function*.

3. Continuum approximations

In this section we show that concepts such as continuous symmetry and the principle of least action may be obtained from combinatorial models as a result of the transition to the limit of large numbers. As an illustration, consider the *open* system described by distribution (6). In the case $M = 2$ (*binomial distribution*), all calculations can be done explicitly. In this case both equivalence relations (4) and (5) coincide in virtue of the equality $k_1 + k_2 = T$. Entropy of (6) for $M = 2$ takes the form

$$S = \ln T! - \ln k_1! - \ln(T - k_1)! + k_1 \ln \alpha_1 + (T - k_1) \ln \alpha_2. \tag{9}$$

Applying the growing terms of Stirling's approximation (2) to this formula we have $S \approx S_{\mathrm{Stirling}} = S_{\mathrm{superlin}} + S_{\log}$, where

$$S_{\mathrm{superlin}} = T \ln T - k_1 \ln \left(\frac{k_1}{\alpha_1} \right) - (T - k_1) \ln \left(\frac{T - k_1}{\alpha_2} \right) \qquad (10)$$

and

$$S_{\log} = \frac{1}{2} \ln \left(\frac{T}{2\pi k_1 (T - k_1)} \right)$$

are superlinear and logarithmic parts of S_{Stirling}, respectively. Since $S_{\log}$ and its derivatives are small for large T and k_1, the maximum of entropy (9) is close to that of its leading part (10). Thus, the approximate extremum point $k_1^* = \alpha_1 T$ is obtained by solving the equation

$$\frac{\partial S_{\mathrm{superlin}}}{\partial k_1} = -\ln \left(\frac{k_1}{\alpha_1} \right) + ln \left(\frac{T - k_1}{\alpha_2} \right) = 0 \iff \frac{k_1}{T - k_1} = \frac{\alpha_1}{1 - \alpha_1}.$$

Further, we can expand S_{Stirling} around the point k_1^*. Retaining terms up to the second order, we obtain the chain of approximations

$$S \approx S_{Stirling} \approx S^* = \left. S_{log} \right|_{k_1 = \alpha_1 T} + \frac{1}{2} \left. \frac{\partial^2 S_{superlin}}{\partial k_1^2} \right|_{k_1 = \alpha_1 T} \times (k_1 - \alpha_1 T)^2,$$

which leads to the final formula

$$S \approx S^* = \ln \sqrt{\frac{1}{2\pi T \alpha_1 \alpha_2}} - \frac{1}{2T \alpha_1 \alpha_2} (k_1 - \alpha_1 T)^2. \qquad (11)$$

3.1   On the origin of continuous symmetries

It is well known [3] that the large numbers asymptotic of the probability distribution of symmetric random walk on the lattice $\mathbb{Z}^d$ is the fundamental solution of the heat equation, also called the *heat kernel*

$$K(t, \vec{x}) = \frac{1}{(4\pi t)^{d/2}} \exp \left(-\frac{x_1^2 + x_2^2 + \cdots + x_d^2}{4t} \right). \qquad (12)$$

Here $t \in \mathbb{R}_+$ and $x_i \in \mathbb{R}$ are the continual substitutes for $T \in \mathbb{N}$ and for the difference $k_i - k_{d+i} \in \mathbb{Z}$, respectively.

This gives an example of the emergence of continuous symmetries from the large numbers approximation. The symmetry group of the integer lattice $\mathbb{Z}^d$ has the structure of the semidirect product $\mathbb{Z}^d \rtimes G_d$. For simplicity, we can drop the normal subgroup $\mathbb{Z}^d$, the subgroup of translations, as inessential for our purposes. The group G_d is isomorphic to the semidirect product $(\mathbb{Z}_2)^d \rtimes \mathsf{S}_d$ or, equivalently, to the wreath product $\mathbb{Z}_2 \wr \mathsf{S}_d$. The size of G_d is equal to $2^d d!$. For example, for the square lattice the group $G_{d=2}$ is the symmetry group of a square — dihedral

group of order 8. On the other hand, approximate expression (12) is symmetric with respect to the **orthogonal group** $\mathsf{O}(d, \mathbb{R})$ with cardinality of continuum.

The **Lorentz symmetries** — at least in $1+1$ dimensions — can also be obtained in a similar way. Let us consider approximation (11) for the entropy of binomial distribution. We introduce the following continuous substitutes

$$
\begin{aligned}
x &:= k_1 - k_2, \\
t &:= T, \\
v &:= \alpha_1 - \alpha_2.
\end{aligned}
\tag{13}
$$

Obviously, $-1 \le v \le 1$. With these substitutions, the approximation of binomial distribution takes the form

$$
P^*(x, t) = \mathrm{e}^{S^*} = \sqrt{\frac{2}{\pi (1 - v^2) t}} \exp \left\{ -\frac{1}{2t} \left(\frac{x - vt}{\sqrt{1 - v^2}} \right)^2 \right\}.
\tag{14}
$$

The continuous variables x, t, and v may be called, respectively, the "space", "time", and "velocity".[1] Expression (14) is the fundamental solution of the equation

$$
\frac{\partial P^*(x, t)}{\partial t} + v \frac{\partial P^*(x, t)}{\partial x} = \frac{(1 - v^2)}{2} \frac{\partial^2 P^*(x, t)}{\partial x^2}.
\tag{15}
$$

This equation is called — depending on interpretation of the function $P^*(x, t)$ — the *heat*, or *diffusion*, or *Fokker-Plank* equation. In the "limit of the speed of light" $|v| = 1$ equation (15) turns into the *wave equation*

$$
\frac{\partial P^*(x, t)}{\partial t} \pm \frac{\partial P^*(x, t)}{\partial x} = 0.
$$

Let us introduce the change of variables: $t = T_H + t'$ and $x = vT_H + x'$. If we assume that $t' \ll T_H$ (i.e., T_H can be thought as a "Hubble time", and t' as a "typical time of observation"), then (14) can be rewritten as

$$
P^*(x, t) = \sqrt{\frac{2}{\pi (1 - v^2) T_H}} \exp \left\{ -\frac{1}{2T_H} \left(\frac{x' - vt'}{\sqrt{1 - v^2}} \right)^2 \right\} + O\left(\frac{t'}{T_H} \right).
$$

The principal part of this expression is "relativistically invariant".

3.2 The least action principle as the principle of selection of the most likely configurations

Let us compare the exact probability distributions with their continuum approximations *within* individual equivalence classes of relation (5).

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[1]In the paper [5], which is devoted to the "Zitterbewegung" effect in the $1 + 1$ dimensional Dirac equation, a "drift velocity" is defined — just like in (13) — as the difference of probabilities of steps in opposite directions. It is shown that this definition leads to the relativistic velocity addition rule: $w = (u + v)/(1 + uv)$.

Exact distributions. In the binomial case, an equivalence class of (5) is defined by fixing the difference $k_1 - k_2 =: X$. We denote the equivalence class of sequences connecting the space-time points $(0,0)$ and (X,T) by $\lambda_{X,T}$. The size of $\lambda_{X,T}$ is equal to

$$N_{X,T} = \frac{T!}{k_1! k_2!} \equiv \frac{T!}{\left(\frac{T+X}{2}\right)! \left(\frac{T-X}{2}\right)!} .$$

The binomial distribution in terms of the variables X and T and parameter v takes the form

$$P(X,T) = \frac{T!}{\left(\frac{T+X}{2}\right)! \left(\frac{T-X}{2}\right)!} \left(\frac{1+v}{2}\right)^{\frac{T+X}{2}} \left(\frac{1-v}{2}\right)^{\frac{T-X}{2}} .$$

Consider an increasing sequence of time instants ("times of observations")

$$\tau = \{T_0 = 0, \ldots, T_{i-1}, T_i, \ldots, T_n = T\}, \quad T_{i-1} < T_i. \tag{16}$$

Let us select trajectories that pass through the sequence of spatial points

$$\chi = \{X_0 = 0, \ldots, X_{i-1}, X_i, \ldots, X_n = X\} \tag{17}$$

corresponding to sequence of times (16). Admissible trajectories must satisfy the inequality $|X_i - X_{i-1}| \le T_i - T_{i-1}$ — "the light cone restriction". According to the *conditional probability* rule, the probability of the trajectory (χ, τ) is equal to

$$\begin{aligned}
P_{\chi,\tau} &= \frac{1}{P(X,T)} \prod_{i=1}^{n} P(X_i - X_{i-1}, T_i - T_{i-1}) \\
&= \frac{\left(\frac{T+X}{2}\right)! \left(\frac{T-X}{2}\right)!}{T!} \prod_{i=1}^{n} \frac{(T_i - T_{i-1})!}{\left(\frac{T_i+X_i}{2} - \frac{T_{i-1}+X_{i-1}}{2}\right)! \left(\frac{T_i-X_i}{2} - \frac{T_{i-1}-X_{i-1}}{2}\right)!} .
\end{aligned}$$

For a given sequence of time instants (16) one can formulate the problem of finding trajectories with maximum probability $P_\tau = \max_\chi P_{\chi,\tau}$. This can be done by searching among all admissible sequences (17). In the general case, there are many distinct trajectories with the same maximum probability, i.e., we do not have here a "deterministic" trajectory.

Continuum approximation. To apply our reasoning to approximate distribution (14), we will consider the time sequence

$$\tau = \{t_0 = t_a, \ldots, t_{i-1}, t_i, \ldots, t_n = t_b\}$$

together with respective sequences of spacial points

$$\chi = \{x_0 = x_a, \ldots, x_{i-1}, x_i, \ldots, x_n = x_b\}. \tag{18}$$

We assume that the time points are equidistant: $t_i - t_{i-1} = \Delta t$, and we will use the notation $\Delta x_i = x_i - x_{i-1}$.

Now the approximate probability of a trajectory (χ, τ) that connects the space-time points (x_a, t_a) and (x_b, t_b) takes the form

$$P_{\chi,\tau}^* = \sqrt{\frac{2}{\pi \left(1 - v^2\right)\left(t_b - t_a\right)}} \exp\left\{-\frac{1}{2\left(t_b - t_a\right)}\left(\frac{x_b - x_a - v\left(t_b - t_a\right)}{\sqrt{1 - v^2}}\right)^2\right\}$$

$$\times \frac{Q_{\chi,\tau}}{A^n} \ ,$$

where

$$A = \sqrt{\frac{\pi\left(1 - v^2\right)\Delta t}{2}}$$

and

$$Q_{\chi,\tau} = \prod_{i=1}^n \exp\left\{-\frac{1}{2\Delta t}\left(\frac{\Delta x_i - v\Delta t}{\sqrt{1 - v^2}}\right)^2\right\} \tag{19}$$

$$= \exp\left\{-\frac{1}{2}\sum_{i=1}^n \left(\frac{\Delta x_i/\Delta t - v}{\sqrt{1 - v^2}}\right)^2 \Delta t\right\}. \tag{20}$$

The summation of factors in (19) over all values $\Delta x_i \in (-\infty, \infty)$ reproduces correct normalization of probabilities for any time slice:

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\Delta t}\left(\frac{\Delta x_i - v\Delta t}{\sqrt{1 - v^2}}\right)^2\right\} \frac{d\left(\Delta x_i\right)}{2} = \sqrt{\frac{\pi\left(1 - v^2\right)\Delta t}{2}} \equiv A.$$

Note that this normalization[1] is an approximation which is incompatible with the "speed of light limitation": $-\Delta t \leq \Delta x_i \leq \Delta t$.

Replacing the sequence of spacial points (18) by a differentiable function $x(t)$ such that $x(t_a) = x_a$, $x(t_b) = x_b$, introducing approximation $\Delta x_i \approx \dot{x}(t)\Delta t$ and taking the limit $n \to \infty$ we can write instead of (20) the formula

$$Q_{\chi,\tau} \approx \exp\left\{-\frac{1}{2}S\left[x(t)\right]\right\},$$

where

$$S\left[x(t)\right] = \int_{t_a}^{t_b} L\,dt = \int_{t_a}^{t_b}\left(\frac{\dot{x}(t) - v}{\sqrt{1 - v^2}}\right)^2 dt \ .$$

Assuming for a while that v depends on t and x, we obtain the following Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \ \Rightarrow \ \ddot{x}\left(1 - v^2\right) + \dot{x}^2 v\frac{\partial v}{\partial x} + 2\dot{x}v\frac{\partial v}{\partial t} - v\frac{\partial v}{\partial x} - \left(1 + v^2\right)\frac{\partial v}{\partial t} = 0.$$

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[1] A similar normalization is one of the cornerstones of the path integral formalism [6].

Clearly, this equation describes "deterministic" trajectories. If we return to the initial assumption that v does not depend on the space-time variables, then the Euler-Lagrange equation reduces to the form

$$\ddot{x}(t) = 0.$$

This equation together with the boundary conditions gives the following formula for the extremals

$$x(t) = \frac{x_b - x_a}{t_b - t_a} t + \frac{x_a t_b - x_b t_a}{t_b - t_a},$$

i.e., the most probable trajectories are straight lines.

4. Combinatorial models of quantum systems

To build models that can reproduce quantum behavior, it is necessary to formulate the basic ingredients of quantum theory in a constructive way (for a more detailed consideration see [7]).

4.1 Constructive core of quantum mechanics

In traditional matrix formulation quantum evolutions are described by unitary operators in a Hilbert space $\mathcal{H}$. Evolution operators U belong to a *unitary representation* of the *continuous* group $\mathrm{Aut}(\mathcal{H})$ of automorphisms of $\mathcal{H}$. To make the problem constructive we should replace the group $\mathrm{Aut}(\mathcal{H})$ by some *finite* group G which should be empirically equivalent to (a subgroup of) $\mathrm{Aut}(\mathcal{H})$.

The theory of quantum computing [8] proves the existence of *finite* sets of universal *quantum gates* that can be combined into unitary matrices which approximate to arbitrary precision any unitary operator. In other words, there exists a *finitely generated* group G_∞ which is a *countable* dense subgroup of the continuous group $\mathrm{Aut}(\mathcal{H})$.

A group G is called *residually finite* [9], if for every $g \in G$, $g \neq \mathbf{1}$, there exists a homomorphism ϕ from G onto a finite group H, such that $\phi(g) \neq \mathbf{1}$. This means that any relation between the elements of G can be modeled by a relation between the elements of a finite group. This can be illustrated by analogy with the widely used in physics technique, when an infinite space is replaced by, for example, a torus whose size is sufficient to hold the data related to a particular problem.

According to the theorem of A.I. Mal'cev [10], every finitely generated group of matrices over any field is residually finite. Thus we have the sequence of transitions from the group with cardinality of *continuum* through a *countable* group to a *finite* group: $\mathrm{Aut}(\mathcal{H}) \xrightarrow{\text{approximation}} \mathsf{G}_\infty \xrightarrow{\text{homomorphism}} \mathsf{G}$.

4.2 Permutations and natural quantum amplitudes

As is well known, any linear representation of a finite group is unitary. Any representation of a finite group is a subrepresentation of some permutation representation (see, e.g., [11–15]). Let U be a representation of G in a K-dimensional Hilbert space

$\mathcal{H}_K$. Then U can be embedded into a permutation representation P of G in an N-dimensional Hilbert space $\mathcal{H}_N$, where $N \geq K$. The representation P is equivalent to an action of G on a set of things $\Omega = \{\omega_1, \ldots, \omega_N\}$ by permutations. In the proper case $N > K$, the embedding has the structure

$$T^{-1}PT = \begin{pmatrix} \mathbf{1} & & \\ & V & \left.\right\} \mathcal{H}_{N-K} \\ & & U \left.\right\} \mathcal{H}_K \end{pmatrix}, \quad \mathcal{H}_N = \mathcal{H}_{N-K} \oplus \mathcal{H}_K, \tag{21}$$

where $\mathbf{1}$ is the trivial one-dimensional representation, mandatory for any permutation representation; V is a subrepresentation, which may be missing. T is a matrix of transition from the basis of the representation P to the basis in which the permutation space $\mathcal{H}_N$ is split into the invariant subspaces $\mathcal{H}_{N-K}$ and $\mathcal{H}_K$. Evolutions in the spaces $\mathcal{H}_K$ and $\mathcal{H}_{N-K}$ are *independent* since both spaces are invariant subspaces of $\mathcal{H}_N$. So we can treat the data in $\mathcal{H}_{N-K}$ as "hidden parameters" with respect to the data in $\mathcal{H}_K$.

A trivial approach would be to set arbitrary (e.g., zero) data in the complementary subspace $\mathcal{H}_{N-K}$. This approach is not interesting since it is not falsifiable by means of standard quantum mechanics. In fact, it leads to standard quantum mechanics *modulo* the empirically unobservable distinction between the "finite" and the "infinite". The only difference is technical: we can replace the linear algebra in the K-dimensional space $\mathcal{H}_K$ by permutations of N things.

A more promising approach requires some changes in the concept of quantum amplitudes. We assume [7, 16, 17] that quantum amplitudes are projections onto invariant subspaces of vectors of multiplicities of elements of the set Ω on which the group G acts by permutations. The vectors of multiplicities

$$|n\rangle = \begin{pmatrix} n_1 \\ \vdots \\ n_N \end{pmatrix} \tag{22}$$

are elements of the *module* $H_N = \mathbb{N}^N$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the semiring of natural numbers. Initially we deal with the natural permutation representation of G in the module H_N. Using the fact that all eigenvalues of any linear representation of a finite group are roots of unity, we can turn the module H_N into a Hilbert space $\mathcal{H}_N$. It is sufficient to add $\mathcal{C}$th roots of unity to the natural numbers to form a *semiring*, which we denote by $\mathbb{N}_{\mathcal{C}}$. The natural number $\mathcal{C}$, called conductor, is (a divisor of) the *exponent* of G, which is defined as the least common multiple of the orders of elements of G. In the case $\mathcal{C} \geq 2$ the *negative integers* can be introduced and the semiring $\mathbb{N}_{\mathcal{C}}$ becomes a *ring of cyclotomic integers*. To complete the conversion of the module H_N into the Hilbert space $\mathcal{H}_N$, we introduce the *cyclotomic field* $\mathbb{Q}_{\mathcal{C}}$ as a field of fractions of the ring $\mathbb{N}_{\mathcal{C}}$.[1] If $\mathcal{C} \geq 3$, then $\mathbb{Q}_{\mathcal{C}}$ is a dense subfield of the field

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[1]By taking into account symmetries of a specific problem, we can use instead of $\mathbb{Q}_{\mathcal{C}}$ some its subfield, an *abelian number field* $\mathcal{F} \leq \mathbb{Q}_{\mathcal{C}}$.

of complex numbers $\mathbb{C}$. In fact, algebraic properties of elements of $\mathbb{Q}_{\mathcal{C}}$ are quite sufficient for all our purposes — for example, complex conjugation corresponds to the transformation $\mathsf{r}^k \to \mathsf{r}^{\mathcal{C}-k}$ for roots of unity — so we can forget the possibility to embed $\mathbb{Q}_{\mathcal{C}}$ into $\mathbb{C}$ (as well as the very existence of the field $\mathbb{C}$).

Thus, we will assume that $\mathcal{H}_{\mathsf{K}}$ in decomposition (21) is a Hilbert space over the field $\mathbb{Q}_{\mathcal{C}}$, and quantum amplitudes are elements of $\mathcal{H}_{\mathsf{K}}$ of the form $|\psi\rangle = \mathrm{Pr}_{\mathsf{U}} |n\rangle$, where Pr_{U} is the projection operator from the module H_{N} onto the space $\mathcal{H}_{\mathsf{K}}$ corresponding to the subrepresentation U in (21).

4.3   Measurements and the Born rule

The general scheme of measurements[1] in quantum mechanics is reduced to the following.

- A partition of the Hilbert space into mutually orthogonal subspaces is given:

$$\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_i \oplus \cdots .$$

  Typically $\mathcal{H}_i$ are eigenspaces of some Hermitian operator A (i.e. $A = A^{\dagger}$) called an "observable".

- There is a measuring device configured to select a state $|\phi\rangle \in \mathcal{H}_i$ of a quantum system.

- A result of a single measurement $= \begin{cases} \text{Yes,} & \text{measuring device responds;} \\ \text{No,} & \text{no response.} \end{cases}$

  In accordance with the *projection postulate*, the output Yes is interpreted as transition of the system into the state $|\phi\rangle$ *after* the measurement.

  If $\mathcal{H}_i$ is an eigenspace of an observable A with eigenvalue a, it is said that the "outcome of the measurement is equal to" a.

- Relative number of Yes in a set of measurements is described by the Born formula.

The Born rule[2] states that the probability to register a particle described by the amplitude $|\psi\rangle$ by an apparatus configured to select the amplitude $|\phi\rangle$ is

$$\mathbf{P}(\phi, \psi) = \frac{|\langle \phi \mid \psi \rangle|^2}{\langle \phi \mid \phi \rangle \langle \psi \mid \psi \rangle} .$$

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[1]To avoid inessential technical complications we consider here only the case of pure states.

[2]There have been many attempts to derive the Born rule from the other physical assumptions — the Schrödinger equation, many-worlds interpretation, etc. However, Gleason's theorem [18] shows that the Born rule is a logical consequence of the very definition of a Hilbert space and has nothing to do with the laws of evolution of physical systems.

In the "finite" background the only reasonable interpretation of probability is the *frequency interpretation*: probability is the ratio of the number of "favorable" combinations to the total number of combinations. So we expect that $\mathbf{P}(\phi, \psi)$ must be a *rational number* if everything is arranged correctly. Thus, in our approach the usual *non-constructive* contraposition — *complex numbers* as intermediate values against *real numbers* as observable values — is replaced by the *constructive* one — *irrationalities* against *rationals*. From the constructive point of view, there is no fundamental difference between irrationalities and constructive complex numbers: both are elements of algebraic extensions.

4.4 Illustration: natural amplitudes and invariant subspaces of permutation representation

Consider the action of the alternating group A_5 on the vertices of the icosahedron. The group A_5 has a *presentation* of the form

$$\mathsf{A}_5 = \left\langle a, b \,\middle|\, a^5 = b^2 = (ab)^3 = \mathbf{1} \right\rangle. \tag{23}$$

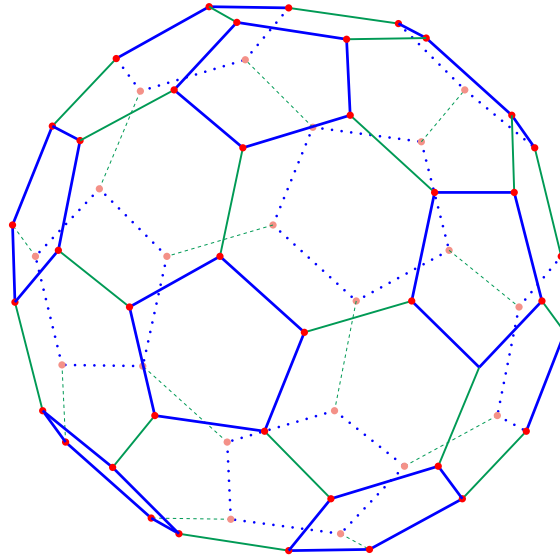The Cayley graph of this presentation is shown in Figure 1.

Figure 1: Cayley graph of A_5 for presentation (23). Pentagons, hexagons and links between adjacent pentagons correspond to the relators a^5, $(ab)^3$ and b^2, respectively.

A_5 has five irreducible representations: the trivial $\mathbf{1}$ and four faithful representations $\mathbf{3}, \mathbf{3'}, \mathbf{4}, \mathbf{5}$; and three *primitive*[1] permutation representations having the following decompositions into the irreducible components: $\underline{\mathbf{5}} \cong \mathbf{1} \oplus \mathbf{4}$, $\underline{\mathbf{6}} \cong \mathbf{1} \oplus \mathbf{5}$, and $\underline{\mathbf{10}} \cong \mathbf{1} \oplus \mathbf{4} \oplus \mathbf{5}$.

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[1] A transitive action of a group on a set is called *primitive* [13], if there is no *non-trivial partition* of the set, invariant under the action of the group.

The action of A_5 on the icosahedron vertices $\Omega = \{1, \ldots, 12\}$ is transitive, but *imprimitive* with the non-trivial partition into the following blocks

$$\{\mid B_1 \mid \cdots \mid B_i \mid \cdots \mid B_6 \mid\} \equiv \{\mid 1, 7 \mid \cdots \mid i, i+6 \mid \cdots \mid 6, 12 \mid\},$$

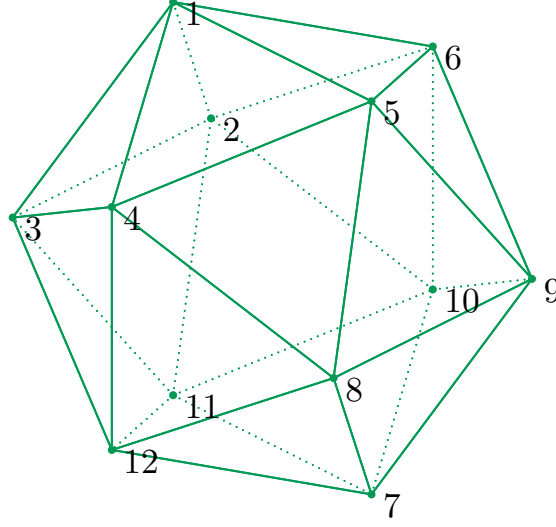assuming the vertex numbering shown in Figure 2. Each block B_i consists of a pair

Figure 2: Icosahedron. Invariant blocks are pairs of opposite vertices.

of opposite vertices of the icosahedron. Permutation representation of the action of A_5 on the icosahedron vertices has the following decomposition into irreducible components

$$\mathrm{T}^{-1}\,(\underline{\mathbf{12}})\,\mathrm{T} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{5}, \tag{24}$$

where T is a matrix of transition from the "permutation" to the "splitting" basis.

Actually there is no necessity to compute transformation matrices like T in (24) explicitly. There is a way [7] to express invariant scalar products in invariant subspaces in terms of easily computable matrices of *orbitals* [14, 15], i.e., orbits of the action of a group G on the Cartesian product $\Omega \times \Omega$.

For the action of A_5 on the set of icosahedron vertices, the matrices of orbitals have the form

$$\mathcal{A}_1 = \mathbb{I}_{12}, \; \mathcal{A}_2 = \begin{pmatrix} 0 & \mathbb{I}_6 \\ \mathbb{I}_6 & 0 \end{pmatrix}, \; \mathcal{A}_3 = \begin{pmatrix} X & Y \\ Y & X \end{pmatrix}, \; \mathcal{A}_4 = \begin{pmatrix} Y & X \\ X & Y \end{pmatrix}, \tag{25}$$

where $\mathbb{I}_n$ is $n \times n$ identity matrix, and

$$X = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

In terms of matrices (25) the invariant bilinear forms (scalar products) corresponding to decomposition (24) take the form

$$\mathcal{B}_{\mathbf{1}} = \frac{1}{12}\left(\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4\right),$$

$$\mathcal{B}_{\mathbf{3}} = \frac{1}{4}\left(\mathcal{A}_1 - \mathcal{A}_2 - \frac{1 + 2r^2 + 2r^3}{5}\mathcal{A}_3 + \frac{1 + 2r^2 + 2r^3}{5}\mathcal{A}_4\right),$$

$$\mathcal{B}_{\mathbf{3'}} = \frac{1}{4}\left(\mathcal{A}_1 - \mathcal{A}_2 + \frac{1 + 2r^2 + 2r^3}{5}\mathcal{A}_3 - \frac{1 + 2r^2 + 2r^3}{5}\mathcal{A}_4\right),$$

$$\mathcal{B}_{\mathbf{5}} = \frac{5}{12}\left(\mathcal{A}_1 + \mathcal{A}_2 - \frac{1}{5}\mathcal{A}_3 - \frac{1}{5}\mathcal{A}_4\right),$$

where r is a 5th primitive root of unity. It is easy to verify that the cyclotomic integer $1 + 2r^2 + 2r^3$ is equal to $-\sqrt{5}$.

Let us consider the scalar products of projections of "natural" vectors. If projections of vectors with natural components $m = (m_1, \ldots, m_{12})^T$ and $n = (n_1, \ldots, n_{12})^T$ onto the invariant subspaces corresponding to $\alpha = \mathbf{1}, \mathbf{3}, \mathbf{3'}, \mathbf{5}$ are Φ_α and Ψ_α, respectively, then $\langle \Phi_\alpha \,|\, \Psi_\alpha \rangle = \langle m \,|\mathcal{B}_\alpha|\, n \rangle$. That is, we have

$$\langle \Phi_{\mathbf{1}} \mid \Psi_{\mathbf{1}} \rangle = \frac{1}{12}\Big(\langle m|\mathcal{A}_1|n\rangle + \langle m|\mathcal{A}_2|n\rangle + \langle m|\mathcal{A}_3|n\rangle + \langle m|\mathcal{A}_4|n\rangle\Big), \qquad (26)$$

$$\langle \Phi_{\mathbf{3}} \mid \Psi_{\mathbf{3}} \rangle = \frac{1}{4}\left(\langle m|\mathcal{A}_1|n\rangle - \langle m|\mathcal{A}_2|n\rangle + \frac{\sqrt{5}}{5}\Big(\langle m|\mathcal{A}_3|n\rangle - \langle m|\mathcal{A}_4|n\rangle\Big)\right),$$

$$\langle \Phi_{\mathbf{3'}} \mid \Psi_{\mathbf{3}} \rangle = \frac{1}{4}\left(\langle m|\mathcal{A}_1|n\rangle - \langle m|\mathcal{A}_2|n\rangle - \frac{\sqrt{5}}{5}\Big(\langle m|\mathcal{A}_3|n\rangle + \langle m|\mathcal{A}_4|n\rangle\Big)\right),$$

$$\langle \Phi_{\mathbf{5}} \mid \Psi_{\mathbf{5}} \rangle = \frac{5}{12}\left(\langle m|\mathcal{A}_1|n\rangle + \langle m|\mathcal{A}_2|n\rangle - \frac{1}{5}\Big(\langle m|\mathcal{A}_3|n\rangle + \langle m|\mathcal{A}_4|n\rangle\Big)\right).$$

Let us give two remarks on these expressions:

- Scalar product (26) can be written as

$$\langle \Phi_{\mathbf{1}} \mid \Psi_{\mathbf{1}} \rangle = \frac{1}{12}\left(m_1 + m_2 + \cdots + m_{12}\right)\left(n_1 + n_2 + \cdots + n_{12}\right).$$

This is the general case: any permutation representation of any group contains the trivial one-dimensional subrepresentation with the scalar product like (26):

$$\langle \Phi_{\mathbf{1}} \mid \Psi_{\mathbf{1}} \rangle = \frac{1}{\mathsf{N}}\left(\sum_{i=1}^{\mathsf{N}} m_i\right)\left(\sum_{i=1}^{\mathsf{N}} n_i\right).$$

The trivial subrepresentation can be interpreted as the "counter of particles", since the linear permutation invariant $\sum_{i=1}^{\mathsf{N}} n_i$ is the total number of elements from Ω in the ensemble described by the vector n.

- The Born probabilities for subrepresentations $\mathbf{3}$ and $\mathbf{3}'$ contain irrationalities that contradicts the frequency interpretation of probability for finite sets. Obviously, this is a consequence of the imprimitivity: one can not move an icosahedron vertex without simultaneous movement of its opposite. To resolve the contradiction, mutually conjugate subrepresentations $\mathbf{3}$ and $\mathbf{3}'$ must be considered together. The scalar product

$$\langle \Phi_{\mathbf{3}\oplus\mathbf{3}'} \mid \Psi_{\mathbf{3}\oplus\mathbf{3}'} \rangle = \frac{1}{2}\Big(\langle m \,|\mathcal{A}_1|\, n \rangle - \langle m \,|\mathcal{A}_2|\, n \rangle \Big)$$

in the six-dimensional subrepresentation $\mathbf{3} \oplus \mathbf{3}'$ always gives rational Born's probabilities for vectors of multiplicities defined as in (22).

4.5 Quantum evolution

In standard quantum mechanics an elementary step of evolution is described by the Schrödinger equation

$$i\frac{\mathrm{d}}{\mathrm{d}t} |\psi\rangle = H |\psi\rangle .$$

In quantum mechanics based on a finite group G a step of evolution has the form

$$|\psi_{t+1}\rangle = U |\psi_t\rangle ,$$

where $U = \mathrm{U}(g)$, $g \in \mathsf{G}$ and U is an unitary representation of G. In this case, there is no need for a Hamiltonian, though, for comparison purposes, it can be introduced by the formula

$$H = i\ln U = \pi\big(\alpha_0\mathbb{I} + \alpha_1 U + \cdots + \alpha_{n-1}U^{n-1}\big) ,$$

where $\mathbb{I}$ is the unit matrix; n is the period of U, i.e. $U^n = \mathbb{I}$; $\alpha_k \in \mathcal{F}$ are easily computable coefficients; $0 \leq k < n$. The energy levels (eigenvalues) of H are $E_k = \dfrac{2\pi k}{n}$. The non-algebraic (transcendental) number π appears here as the result of summation of infinite series — the natural logarithm is essentially an infinite construct.

Note that a single unitary evolution is physically trivial, as it describes only a change of coordinates ("rotation") in a Hilbert space. Namely, for the evolution of a pair of vectors $|\phi_T\rangle = U |\phi_0\rangle$, $|\psi_T\rangle = U |\psi_0\rangle$, we have

$$\langle \phi_T \,|\, \psi_T \rangle = \big\langle \phi_0 \,\big|U^\dagger U\big|\, \psi_0 \big\rangle \equiv \langle \phi_0 \,|\, \psi_0 \rangle .$$

This means that a single deterministic evolution can not provide physically observable effects. Thus, a collection of different evolutions is needed. Suppose that the operators of evolution belong to a unitary representation U of a group G. Then two different evolutions U and V can be represented as $U = \mathrm{U}(g_1 g_2 \cdots g_T)$ and $V = \mathrm{U}(f_1 f_2 \cdots f_T)$, where $g_1, \cdots, g_T; f_1, \cdots, f_T \in \mathsf{G}$. These evolutions provide a nontrivial physical effect if

$$\langle \phi_T \,|\, \psi_T \rangle = \big\langle \phi_0 \,\big|V^\dagger U\big|\, \psi_0 \big\rangle \neq \langle \phi_0 \,|\, \psi_0 \rangle \iff V^\dagger U = \mathrm{U}\big(f_T^{-1} \cdots f_1^{-1} g_1 \cdots g_T\big) \neq \mathbb{I}$$

or

$$h = f_T^{-1} \cdots f_1^{-1} g_1 \cdots g_T \neq \mathbf{1},$$

where $\mathbf{1}$ is the identity of G. The expression $h = f_T^{-1} \cdots f_1^{-1} g_1 \cdots g_T$ is called the *holonomy* at the point T of principal G-connection. In differential geometry, infinitesimal analogue of holonomy is called the *curvature* of the corresponding connection. As is well known, all fundamental physical forces are represented in the gauge theories as curvatures of appropriate connections.

Observations support the view that fundamental indeterminism is really the *modus operandi* of nature.[1] In our view this indeterminism arises from a fundamental impossibility to trace the identity of indistinguishable objects during their evolution. Hermann Weyl discussed this issue in detail in [19]. More formally, identification of objects at different time points is provided by a connection (parallel transport).

In our setting we consider dynamical system as a fiber bundle $(\mathsf{W}, \mathcal{T}, \Omega, \mathsf{G}, \tau)$ over discrete time $\mathcal{T} = [0, 1, \ldots, T]$, where *typical fiber* $\Omega = \{\omega_1, \ldots, \omega_\mathsf{N}\}$ is *canonical set* of states, *structural group*

$$\mathsf{G} = \{\mathsf{g}_1, \ldots, \mathsf{g}_m, \ldots, \mathsf{g}_\mathsf{M}\} \leq \mathrm{Sym}\,(\Omega) \cong \mathsf{S}_\mathsf{N} \qquad (27)$$

is the *group of symmetries* of states, τ is a projection $\mathsf{W} \to \mathcal{T}$.

Connection (*parallel transport*) $\chi_{t_1 t_2} \in \mathsf{G}$ defines isomorphism between the fibers at different times of observations:

$$\Omega_{t_2} = \Omega_{t_1} \chi_{t_1 t_2}\,.$$

There is no objective way to choose the "correct" value for the connection. *A priori*, any element of G may serve as $\chi_{t_1 t_2}$. In continuous gauge theories, the gauge fields (fields of connections) are determined from the principle of least action using Lagrangians chosen for different reasons. For example, in the case of the Yang-Mills theory (covering also the case of Maxwell equations), the Lagrangian $L_{\mathrm{YM}} = \mathrm{Tr}\,[F \wedge \star F]$ is used, where F is the curvature form of a gauge connection, $\star$ denotes the *Hodge conjugation*. Analysis of the structure of L_{YM} in the discrete approximation [20] shows that it can be expressed in terms of traces of the fundamental representation of holonomies of a gauge group.

4.6 Combinatorial models of gauge and quantum evolution

Consider a simple combinatorial model involving random choice of the rules for identification of states of dynamical systems at different points of time. The time of our model is the sequence $\mathcal{T} = [0, 1, \ldots, t-1, t, \ldots, T]$. A parallel transport

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[1] There are persistent attempts to develop a deterministic version of quantum mechanics, as if determinism were a "synthetic *a priori* judgment" — an inevitable (though not deducible from logic alone) necessity. However, since the time of Kant up to now there are no convincing evidences of the very existence of judgments of this kind. More likely, the belief in determinism is a mental habit formed by long macroscopic experience.

connecting the initial and the final time points can be decomposed into the product of elementary steps:

$$\chi_{0,1} \cdots \chi_{t-1,t} \cdots \chi_{T-1,T} . \tag{28}$$

We assume that any elementary step is an element of group (27) with probability independent of time:

$$\mathbf{P}\left(\chi_{t-1,t} = \mathsf{g}_m \in \mathsf{G}\right) = \alpha_m, \quad \sum_{m=1}^{\mathsf{M}} \alpha_m = 1 .$$

All possible paths (28) form the set of microstates U. The microstate corresponding to (28) is the sequence $u = \mathsf{g}_{m_1}, \ldots, \mathsf{g}_{m_t}, \ldots, \mathsf{g}_{m_T}$, where $\mathsf{g}_{m_t} = \chi_{t-1,t}$. Its probability is the product $p_u = \alpha_{m_1} \cdots \alpha_{m_t} \cdots \alpha_{m_T}$.

We can define a natural equivalence relation on U as the triviality of the holonomy of a pair of paths:

$$\chi_{0,1} \cdots \chi_{T-1,T} \sim \gamma_{0,1} \cdots \gamma_{T-1,T} \iff \gamma_{T-1,T}^{-1} \cdots \gamma_{0,1}^{-1} \chi_{0,1} \cdots \chi_{T-1,T} = \mathbf{1} .$$

This equivalence allows us to define M macrostates $\lambda_1, \ldots, \lambda_{\mathsf{M}}$. Statistical evolution of all the macrostates can be calculated simultaneously by a simple algorithm. The distribution of the macrostates at the moment T is the following element of the group algebra

$$A_T \equiv P_{\lambda_1} \mathsf{g}_1 + P_{\lambda_2} \mathsf{g}_2 + \cdots + P_{\lambda_{\mathsf{M}}} \mathsf{g}_{\mathsf{M}} = \left(\alpha_1 \mathsf{g}_1 + \alpha_2 \mathsf{g}_2 + \cdots + \alpha_{\mathsf{M}} \mathsf{g}_{\mathsf{M}}\right)^T .$$

The algorithm of *binary exponentiation* computes this expression by performing $O\left(\log T\right)$ multiplications. In simple cases the probabilities P_{λ_m} can be written explicitly, e.g., for the cyclic group $\mathsf{G} = \mathbb{Z}_{\mathsf{M}}$ we have (assuming $\mathsf{g}_1 = \mathbf{1}$)

$$P_{\lambda_m} = \sum_{\substack{k_1 + k_2 + \cdots + k_{\mathsf{M}} = T \\ k_2 + 2k_3 + \cdots + (\mathsf{M}-1)k_{\mathsf{M}} \equiv m \pmod{\mathsf{M}}}} \frac{T!}{k_1! k_2! \cdots k_{\mathsf{M}}!} \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_{\mathsf{M}}^{k_{\mathsf{M}}} .$$

Having a representation U of the group G in a Hilbert space $\mathcal{H}$ we can associate with gauge evolution (28) the quantum evolution:

$$|\psi_T\rangle = \mathsf{U}\left(\chi_{T-1,T}^{-1} \cdots \chi_{t-1,t}^{-1} \cdots \chi_{0,1}^{-1}\right) |\psi_0\rangle , \quad \psi_0, \psi_T \in \mathcal{H} .$$

Simulation of many important features of quantum behavior requires models involving spatial structures explicitly. The set of states of a system with space is the set of functions

$$\Omega = \Sigma^{\mathsf{X}}, \tag{29}$$

where

$$\mathsf{X} = \{\mathsf{x}_1, \ldots, \mathsf{x}_\mathsf{n}\}$$

is a *space*, and

$$\Sigma = \left\{ \sigma_1, \ldots, \sigma_{|\Sigma|} \right\}$$

is a set of *local states*. Having the groups of *spatial*

$$\mathsf{F} = \left\{ \mathsf{f}_1, \ldots, \mathsf{f}_{|\mathsf{F}|} \right\} \leq \mathrm{Sym}\,(\mathrm{X})$$

and *internal*

$$\Gamma = \left\{ \gamma_1, \ldots, \gamma_{|\Gamma|} \right\} \leq \mathrm{Sym}\,(\Sigma)$$

symmetries we can construct a symmetry group of the whole system . This group, having a structure of the *wreath product*

$$\mathsf{G} = \Gamma \wr_{\mathrm{X}} \mathsf{F} \cong \Gamma^{\mathrm{X}} \rtimes \mathsf{F},$$

acts on the set Ω given by formula (29).

It is worth to say a few words about the most common quantum models with spatial structures: *quantum cellular automata* [21] and *quantum walks* [22]. It is proved that models of both types are able to perform any quantum computation, i.e., they can simulate quantum Turing machines.

The Hilbert space of a **quantum cellular automaton** has the form

$$\mathcal{H} = \mathcal{H}_{\Sigma}^{\otimes \mathrm{X}},$$

where X is usually a d-dimensional lattice: $\mathbb{Z}^d$ or its finite counterpart $\mathbb{Z}_N^d$ (one can also take an arbitrary regular graph as a lattice X); $\mathcal{H}_{\Sigma}$ is a Hilbert space associated with a set Σ of local states of sites $x \in \mathrm{X}$. It is assumed that there is a local update rule U_x, which is a unitary operator acting on the Hilbert space $\mathcal{H}_{\Sigma}^{\otimes \mathcal{N}_x}$, where $\mathcal{N}_x$ is a neighborhood of the point x. Since the neighborhoods of different points may intersect, some effort should be made to ensure global unitary. To provide the required compatibility several different definitions of quantum cellular automata were proposed. For properly defined automaton the local updates can be combined into a unitary operator U on $\mathcal{H}$ that describes an elementary step of evolution of the whole system. Then the evolution of the system is defined by the operator U^T.

The spatial structure in a model of **quantum walk** is a k-regular graph X. In the most usual case $k = 2$, the space X is taken to be either $\mathbb{Z}$ or $\mathbb{Z}_N$. Let $\mathcal{H}_{\mathrm{X}}$ be the Hilbert space spanned by the vertices of X. The construction of a quantum walk uses also an auxiliary k-dimensional Hilbert space $\mathcal{H}_C$, the "coin space", and a fixed unitary "coin operator" C acting on $\mathcal{H}_C$. A typical coin operator in the case $k > 2$ is the *Grover coin* (*Grover's diffusion operator*):

$$G = 2 \left| \psi \right\rangle \left\langle \psi \right| - \mathbb{I}_k, \quad \text{where} \quad \left| \psi \right\rangle = \frac{1}{\sqrt{k}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

In the case of one-dimensional quantum walk ($k = 2$) many different one-qubit gates, like the *Hadamard gate* etc., are used. In particular, the Grover coin G

coincides with the *Pauli-X gate* at $k = 2$. The Hilbert space of the whole system is the product

$$\mathcal{H} = \mathcal{H}_{\mathrm{X}} \otimes \mathcal{H}_C.$$

Roughly speaking, the coin operator C "selects directions of spatial shifts". The spatial shifts are performed by an unitary *shift operator* S acting on $\mathcal{H}_{\mathrm{X}}$ at conditions given by the coin C. T steps of evolution of the system are performed by the transformation U^T, where U is the following combination of the coin and shift operators

$$U = S \left(\mathbb{I}_k \otimes C \right).$$

5. Summary

Starting with the idea that any problem that has a meaningful empirical content can be formulated in constructive finite terms, we consider the possibility of derivation of many important elements of physical theories in the framework of discrete combinatorial models. We show that such concepts as continuous symmetries, the principle of least action, Lagrangians, deterministic evolution equations can be obtained by applying the large number approximation to expressions for sizes of certain equivalence classes of combinatorial structures.

We adhere to the view that quantum behavior can be explained by the fundamental impossibility to trace identity of indistinguishable objects in the process of their evolution. Gauge connection is that structure which provides the identity: that is why the gauge fields are so important in quantum theory.

Using general mathematical arguments we show that any quantum problem can be reduced to permutations. Quantum interferences are phenomena observed in invariant subspaces of permutation representations and expressed in terms of permutation invariants. In particular, this approach gives an immediate explanation for the appearance of complex numbers and unitarity in the formalism of quantum theory.

We consider some approaches to the construction of discrete models of quantum behavior and related models describing evolution of gauge connections.

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